

# A Universal upper bound on Graph Diameter based on Laplacian Eigenvalues

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## Abstract

We prove that the diameter of any unweighted connected graph  $G$  is  $O(k \log n / \lambda_k)$ , for any  $k \geq 2$ . Here,  $\lambda_k$  is the  $k$  smallest eigenvalue of the normalized laplacian of  $G$ . This solves a problem posed by Gil Kalai.

## 1 Introduction

Let  $G = (V, E)$  be a connected, undirected and unweighted graph, and let  $d(v)$  be the degree of vertex  $v$  in  $G$ . Let  $D$  be the diagonal matrix of vertex degrees and  $A$  be the adjacency matrix of  $G$ . The *normalized laplacian* of  $G$  is the matrix  $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ , where  $I$  is the identity matrix. The matrix  $\mathcal{L}$  is positive semi-definite. Let

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq 2$$

be the eigenvalues of  $\mathcal{L}$ . For any pair of vertices  $u, v \in G$ , we define their distance,  $\text{dist}(u, v)$ , to be the length of the shortest path connecting  $u$  to  $v$ . The diameter of the graph  $G$  is the maximum distance between all pairs of vertices, i.e.,

$$\text{diam}(G) := \max_{u, v} \text{dist}(u, v).$$

The following question is asked by Gil Kalai in a personal communication [Kal12]. Is it true that for any connected graph  $G$ , and any  $k \geq 2$ ,  $\text{diam}(G) = O(k \log(n) / \lambda_k)$ . We remark that for  $k = 2$ , the conjecture is already known to hold, since the mixing time of the lazy random walk on  $G$  is  $O(\log n / \lambda_2)$ . Therefore, this conjecture can be seen as a generalization of the  $k = 2$  case.

In this short note we answer his question affirmatively and we prove the following theorem

**Theorem 1.1.** *For any unweighted, connected graph  $G$ , and any  $k \geq 2$ ,*

$$\text{diam}(G) \leq \frac{48k \log n}{\lambda_k}.$$

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Our proof uses the easy direction of the higher order cheeger inequalities (see e.g. [LOT12]). For a set  $S \subseteq V$ , let  $E(S, \bar{S}) := \{\{u, v\} : |\{u, v\} \cap S| = 1\}$  be the set of edges with exactly one endpoint in  $S$ , and let  $N(S)$  be the set of neighbors of the set  $S$ . Let  $\text{vol}(S) := \sum_{v \in S} d(v)$  be the volume of the set  $S$ , and let

$$\phi(S) := \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

be the conductance of  $S$ .

Let  $\phi_k(G)$  be the worst conductance of any  $k$  disjoint subsets of  $V$ , i.e.,

$$\phi_k(G) := \min_{\text{disjoint } S_1, S_2, \dots, S_k} \max_{1 \leq i \leq k} \phi(S_i).$$

The following theorem is proved in [LOT12]; it shows that for any graph  $G$ ,  $\phi_k(G)$  is well characterized by the  $\lambda_k$ .

**Theorem 1.2** (Lee et al. [LOT12]). *For any graph  $G$ , and any  $k \geq 2$ ,*

$$\frac{\lambda_k}{2} \leq \phi_k(G) \leq O(k^2) \sqrt{\lambda_k}.$$

We use the left side of the above inequality (a.k.a. easy direction of higher order cheeger inequality) to prove [Theorem 1.1](#).

## 2 Proof

In this section we prove [Theorem 1.1](#). We construct  $k$  disjoint sets  $S_1, \dots, S_k$  such that for each  $1 \leq i \leq k$ ,  $\phi(S_i) \leq O(k \log n / \text{diam}(G))$ , and then we use [Theorem 1.2](#) to prove the theorem.

First, we find  $k+1$  vertices  $v_0, \dots, v_k$  such that the distance between each pair of the vertices is at least  $\text{diam}(G)/2k$ . We can do that by taking the vertices  $v_0$  and  $v_k$  to be at distance  $\text{diam}(G)$ . Then, we consider a shortest path connecting  $v_0$  to  $v_k$  and take equally spaced vertices on that path.

For a set  $S \subseteq V$ , and radius  $r \geq 0$  let

$$B(S, r) := \{v : \min_{u \in S} \text{dist}(v, u) \leq r\}$$

be the set of vertices at distance at most  $r$  from the set  $S$ . If  $S = \{v\}$  is a single vertex, we abuse notation and use  $B(v, r)$  to denote the ball of radius  $r$  around  $v$ . For each  $i = 0, \dots, k$ , consider the ball of radius  $\text{diam}(G)/6k$  centered at  $v_i$ , and note that all these balls are disjoint. Therefore, at most one of them can have a volume of at least  $\text{vol}(V)/2$ . Remove that ball from consideration, if present. So, maybe after renaming, we have  $k$  vertices  $v_1, \dots, v_k$  such that the balls of radius  $\text{diam}(G)/6k$  around them,  $B(v_1, \text{diam}(G)/6k), \dots, B(v_k, \text{diam}(G)/6k)$ , are all disjoint and all contain at most a mass of  $\text{vol}(V)/2$ .

The next claim shows that for any vertex  $v_i$  there exists a radius  $r_i < \text{diam}(G)/6k$  such that  $\phi(B(v_i, r_i)) \leq 24k \log n / \text{diam}(G)$ .

**Claim 2.1.** *For any vertex  $v \in V$  and  $r > 0$ , if  $\text{vol}(B(v, r)) \leq \text{vol}(V)/2$ , then for some  $0 \leq i < r$ ,  $\phi(B(v, i)) \leq 4 \log n / r$ .*

*Proof.* First observe that for any set  $S \subseteq V$ , with  $\text{vol}(S) \leq \text{vol}(V)/2$ ,

$$\text{vol}(B(S, 1)) = \text{vol}(S) + \text{vol}(N(S)) \geq \text{vol}(S) + |E(S, \overline{S})| = \text{vol}(S)(1 + \phi(S)) \quad (1)$$

where the inequality follows from the fact that each edge  $\{u, v\} \in E(S, \overline{S})$  has exactly one endpoint in  $N(S)$ , and the last equality follows from the fact that  $\text{vol}(S) \leq \text{vol}(V)/2$ . Now, since  $B(v, r) \leq \text{vol}(V)/2$ , by repeated application of (1) we get,

$$\begin{aligned} \text{vol}(B(v, r)) &\geq \text{vol}(B(v, r-1))(1 + \phi(B(v, r-1))) \geq \dots \geq \prod_{i=0}^{r-1} (1 + \phi(B(v, i))) \\ &\geq \exp\left(\frac{1}{2} \sum_{i=0}^{r-1} \phi(B(v, i))\right). \end{aligned}$$

where the last inequality uses the fact that  $\phi(S) \leq 1$  for any set  $S \subseteq V$ . Since  $G$  is unweighted,  $\text{vol}(B(v, r)) \leq \text{vol}(V) \leq n^2$ . Therefore, by taking logarithm from both sides of the above inequality we get,

$$\sum_{i=0}^{r-1} \phi(B(v, i)) \leq 2 \log(\text{vol}(B(v, r))) \leq 4 \log n.$$

Therefore, there exists  $i < r$  such that  $\phi(B(v, i)) \leq 4 \log n / r$ .  $\square$

Now, for each  $1 \leq i \leq k$ , let  $S_i := B(v_i, r_i)$ . Since  $r_i < \text{diam}(G)/6k$ ,  $S_1, \dots, S_k$  are disjoint. Furthermore, by the above claim  $\phi(S_i) \leq 24k \log n / \text{diam}(G)$ . Therefore,  $\phi_k(G) \leq 24k \log n / \text{diam}(G)$ . Finally, using [Theorem 1.2](#), we get

$$\lambda_k \leq 2\phi_k(G) \leq \frac{48k \log n}{\text{diam}(G)}.$$

This completes the proof of [Theorem 1.1](#).

## References

- [Kal12] Gil Kalai. Personal Communication, 2012. [1](#)
- [LOT12] James R. Lee, Shayan Oveis Gharan, and Luca Trevisan. Multi-way spectral partitioning and higher-order cheeger inequalities. In *STOC*, pages 1117–1130, 2012. [2](#)